Minimal Surface Theory: The Mathematics of Soap Films

Awn Umar

BSc Mathematics and Computer Science

University of Bristol

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1 Introduction

A soap film is represented by a kind of surface that has zero average "curvature" at all points: a **minimal surface**. We begin by introducing the foundational mathematics of differential geometry followed by what it means for a surface to be minimal. We explore the connection between soap films and minimal surfaces and then see how complex analysis can be used to create them. Finally we render some examples using Maple.

This project relies extensively on [Opr00], [HD10] and [Mor09], as well as others. A full list of sources is given in the bibliography. I could not have produced this work without the help of Dr Ivor McGillivray whose patient explanations were invaluable to my understanding.

2 Curves

Definition 2.1. A **curve** α is a 1-dimensional object in n-dimensional space. We will focus on curves in 3 dimensions. We can represent a curve as a differentiable map:

$$\begin{aligned}\alpha : \mathbb{R} \supseteq I &\to \mathbb{R}^3 \\ t &\mapsto (x(t), y(t), z(t))\end{aligned}$$

where I is an interval in the real line and each of x, y, z are differentiable over I.

Definition 2.2. A curve α is **regular** if $\forall\, t \in I$,

$$\alpha'(t) = \frac{d\alpha}{dt}(t) \neq 0$$

Definition 2.3. The **tangent vector** T of a regular curve is the unit velocity at that point:

$$T = \frac{\alpha'(t)}{|\alpha'(t)|}$$

If α is not regular then $\exists\, t \in I$ such that $\alpha'(t) = 0$ and so T is undefined.

Definition 2.4. The **arc length** s along the curve α between $t = a$ and $t = b$ is

$$\begin{aligned}s &= \int_a^b |\alpha'(t)|\, dt \\ &= \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt\end{aligned}$$

Then by the fundamental theorem of calculus,

$$\frac{ds}{dt} = |\alpha'(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$$

Definition 2.5. A curve $\alpha(s)$ is **parametrised by arc length** if

$$s = \int_0^s |\alpha'(t)|\, dt$$

Such a parametrisation has the useful property that the length of $\alpha(s)$ for $a \leq s \leq b$ is simply $b - a$. We can see that

$$s = \int_0^s |\alpha'(t)|\, dt \iff |\alpha'(t)| = \frac{ds}{dt} = 1$$

Imagine α is describing the position of a particle. Then α is parametrised by arc length precisely when the particle is moving at unit speed.

Definition 2.6. The **curvature** κ of a curve is

$$\kappa = \left|\frac{dT}{ds}\right| = \left|\frac{dT}{dt}\frac{dt}{ds}\right| = \left|\frac{\alpha''(t)}{|\alpha'(t)|}\right|$$

This gives the rate of change of the angle which neighbouring tangents make with the tangent vector at some point t.

If α is parametrised by arc length this simplifies to

$$\kappa = \left|\frac{dT}{dt}\right| = |\alpha''(t)|$$

Lemma 2.7. The curvature vector is orthogonal to the tangent vector.

Proof. Since T is a unit vector, we have that $T \cdot T = 1$. Therefore,

$$\frac{d}{ds}(T \cdot T) = 0 \implies T \cdot T' = 0$$

□

Definition 2.8. The **radius of curvature** R is

$$R = \frac{1}{\kappa(t)}$$

This gives the radius of the osculating circle on α at some t. Equivalently, it is the radius of the circular arc that best approximates α at t.

If κ is large, the curve is very tight and so R is small. Conversely if κ is small, the curve is gradual and so R is big.

Example 2.9. Consider the curve given by

$$\alpha(t) = \left(\frac{1}{2}, \frac{1}{4}\cos t, \frac{1}{4}\sin t\right)$$

for $t \in \mathbb{R}$. This is a circle of radius $1/4$ in the yz-plane at $x = 1/2$. We compute

$$\begin{aligned} \alpha'(t) = \left(0, -\frac{1}{4}\sin t, -\frac{1}{4}\cos t\right) &\implies |\alpha'(t)| = \frac{1}{4} \\ \implies T = (0, -\sin t, -\cos t) \end{aligned}$$

and so α is regular and is not parametrised by arc length. The curvature is

$$\frac{dT}{ds} = (0, -4\cos t, 4\sin t) \implies \kappa = 4$$

Then the radius of curvature $R = 1/4$, which is precisely the radius of the circle described by α.

The arc length of α is given by

$$\begin{aligned} s &= \int_a^b |\alpha'(t)|\, dt \\ &= \int_a^b \frac{1}{4}\, dt \\ &= \frac{t}{4}\Big|_a^b \end{aligned}$$

If the input increases by 1, the corresponding change to the output is

$$\frac{t}{4}\Big|_a^{a+1} = \frac{a+1}{4} - \frac{a}{4} = \frac{1}{4}$$

We can reparametrise α by arc length by accounting for this difference:

$$\begin{aligned} \alpha_s(t) &= \left(\frac{1}{2}, \frac{1}{4}\cos 4t, \frac{1}{4}\sin 4t\right) \\ \implies \alpha_s'(t) &= (0, -\sin 4t, \cos 4t) \implies |\alpha_s'| = 1 \end{aligned}$$

Definition 2.10. The **principle unit normal vector** $\widehat{N}$ of a curve is

$$\widehat{N} = \frac{dT/ds}{|dT/ds|}$$

This is the the normalised curvature vector.

Definition 2.11. The **unit binormal vector** $\widehat{B}$ of a curve is

$$\widehat{B} = T \times \widehat{N}$$

It is orthogonal to both the tangent vector T and the normal vector $\widehat{N}$.

Definition 2.12. The **torsion** τ of a curve $\alpha(t)$ is analogous to curvature except it measures how quickly the curve twists out of the plane of curvature. It is given by

$$\tau = -\widehat{N} \cdot \frac{d\widehat{B}}{ds} = -\widehat{N} \cdot \frac{d\widehat{B}/dt}{|\alpha'(t)|}$$

Remark 2.13. If $\tau = 0$ at all points, α is a plane curve. Otherwise the sign of τ gives the direction of rotation.

Definition 2.14. The **Frenet–Serret frame** Λ is the collection

$$\Lambda = \{T, \widehat{N}, \widehat{B}\}$$

By orthogonality, $span\{\Lambda\} = \mathbb{R}^3$.

Theorem 2.15. The **Frenet–Serret formulae** describe the motion of a curve in 3 dimensional space.

$$\frac{dT}{ds} = \kappa \widehat{N} \tag{1}$$

$$\frac{d\widehat{N}}{ds} = -\kappa T + \tau \widehat{B} \tag{2}$$

$$\frac{d\widehat{B}}{ds} = -\tau \widehat{N} \tag{3}$$

In other words, they describe how the vectors in the Frenet-Serret frame change with respect to arc length.

Remark 2.16. Equations (1) and (3) say that T' and $\widehat{B}'$ lie in the span of the normal vector $\widehat{N}$.

Equation (2) says that $\widehat{N}' \in span\{T, \widehat{B}\}$. In other words, $\widehat{N}'$ can be expressed as a linear combination of T and $\widehat{B}$, weighted by the curvature κ and the torsion τ.

Example 2.17. Consider the curve $\alpha(t) = (\cos t, \sin t, t)$ shown in fig. 1. This traces out a circular spiral orbiting the z-axis at unit distance.

Constructing the members of Λ gives

$$T = \Big(-\frac{1}{\sqrt{2}} \sin t, \frac{1}{\sqrt{2}} \cos t, \frac{1}{\sqrt{2}} \Big)$$
$$\widehat{N} = (-\cos t, -\sin t, 0)$$
$$\widehat{B} = \Big(\frac{1}{\sqrt{2}} \sin t, -\frac{1}{\sqrt{2}} \cos t, \frac{1}{\sqrt{2}} \Big)$$

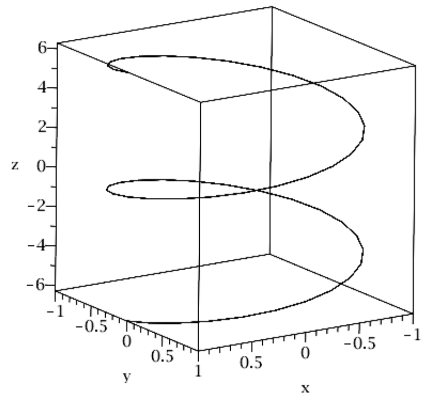


Figure 1: Three dimensional plot of the helix α over $t \in [-2\pi, 2\pi]$.

The curvature and torsion are given by

$$\frac{dT}{ds} = \Big(-\frac{1}{2}\cos t, -\frac{1}{2}\sin t, 0\Big) \implies \kappa = \frac{1}{2} \implies R = 2$$
$$\frac{d\widehat{B}}{ds} = \Big(\frac{1}{2}\cos t, \frac{1}{2}\sin t, 0\Big) \implies \tau = \frac{1}{2}$$

The rate of change of the normal vector with respect to arc length is

$$\frac{d\widehat{N}}{ds} = \Big(\frac{1}{\sqrt{2}}\sin t, -\frac{1}{\sqrt{2}}\cos t, 0\Big)$$

These computations are consistent with the Frenet–Serret formulae.

3 Surfaces

Definition 3.1. A **surface** S is a 2-dimensional object in n-dimensional space. We will focus on surfaces in 3 dimensions. A surface can be represented by a map:

$$\phi : \mathbb{R}^2 \supseteq D \to S \subseteq \mathbb{R}^3$$
$$(u, v) \mapsto (x(u, v), y(u, v), z(u, v))$$

Definition 3.2. The **velocity vectors** of a surface parametrised by $\phi(u, v)$ are

$$\phi_u = \frac{\partial \phi}{\partial u} = \Big(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\Big)$$

$$\phi_v = \frac{\partial \phi}{\partial v} = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right)$$

Definition 3.3. A surface S is **regular** [Car00] if it has the following properties:

1. **Smoothness**: if $\phi = (x, y, z)$ then the functions x, y, z have continuous partial derivatives of all orders.
2. ϕ is a **homeomorphism**. Since ϕ is continuous by condition 1, this means that ϕ also has an inverse $\phi^{-1} : \phi(D) \to D$ that is smooth.
3. **Regularity**: at each $p \in S$ the cross product $\phi_u \times \phi_v$ is non-zero. In other words, the velocity vectors are linearly independent.

Definition 3.4. The **tangent space** T_pS is the set of points reachable by a linear combination of the two velocity vectors at some point $\phi(p)$ on the surface. On a regular surface (by condition 3) the tangent space is a plane.

$$T_pS = span\{\phi_u(p), \phi_v(p)\}$$

Definition 3.5. The **unit surface normal** vector on a regular surface is

$$N = \frac{\phi_u \times \phi_v}{|\phi_u \times \phi_v|}$$

3.1 Constructions

3.1.1 Monge

The Monge parametrisation—named after the French mathematician Gaspard Monge—describes a surface on the z-axis as a function over the 2-dimensional xy-plane.

$$\phi(x, y) = (x, y, f(x, y))$$

Where f is a smooth function. The velocity vectors are

$$\begin{aligned} \phi_x &= (1, 0, \partial_x f) \\ \phi_y &= (0, 1, \partial_y f) \end{aligned}$$

Which are indeed linearly independent:

$$\phi_x \times \phi_y = \begin{vmatrix} i & j & k \\ 1 & 0 & \partial_x f \\ 0 & 1 & \partial_y f \end{vmatrix} = (-\partial_x f, -\partial_y f, 1) \neq 0$$

This formulation yields some useful expressions but it is limited to situations where the surface does not turn back on itself so much that $f(x, y)$ is undefined. The domain can be limited so that the parametrisation is only defined for local regions where this is true.

3.1.2 Spherical

A nice parametrisation for a sphere is the following (from [Opr00]). Let $S \subset \mathbb{R}^3$ be a sphere of radius R centered at the origin. Let

$$D = \Big\{(u,v) \in \mathbb{R}^2 \mid u \in [0, 2\pi), v \in \Big(-\frac{\pi}{2}, \frac{\pi}{2}\Big)\Big\}$$

Then S is parametrised by

$$\begin{aligned} &\phi : D \to S \\ &(u,v) \mapsto (R\cos u \cos v, R\sin u \cos v, R\sin v) \end{aligned}$$

Indeed each component of ϕ is smooth, and the velocity vectors are linearly independent:

$$\begin{aligned} &\phi_u \times \phi_v = (R^2 \cos u \cos^2 v, R^2 \sin u \cos^2 v, R^2 \sin v \cos v) \\ \implies &|\phi_u \times \phi_v| = R^2 \cos v \neq 0 \iff R \neq 0 \end{aligned}$$

The last two-way implication is valid because the values of v in the parametrisation are restricted so that $\cos v \neq 0$.

3.1.3 Revolution

A 3 dimensional surface can be generated by taking a 2 dimensional curve and rotating it about some axis.

Suppose $\alpha(t) = (g(t), h(t), 0)$ is a smooth, regular curve in the xy-plane. Rotating it around the x-axis yields the following parametrisation for the **surface of revolution** S:

$$\phi(t, \theta) = (g(t), h(t)\cos\theta, h(t)\sin\theta)$$

Notice that $\alpha(t) = \phi(t, 0)$, i.e. α is the restriction of ϕ to $\theta = 0$.

Let $g_t = \partial_t g$ and $h_t = \partial_t h$. The velocity vectors are

$$\begin{aligned} \phi_t &= (g_t, h_t \cos\theta, h_t \sin\theta) \\ \phi_\theta &= (0, -h\sin\theta, h\cos\theta) \end{aligned}$$

Consider their cross product:

$$\phi_t \times \phi_\theta = \begin{vmatrix} i & j & k \\ g_t & h_t \cos\theta & h_t \sin\theta \\ 0 & -h\sin\theta & h\cos\theta \end{vmatrix}$$

$$= (hh_t, -g_t h\cos\theta, -g_t h\sin\theta)$$
$$\implies |\phi_t \times \phi_\theta|^2 = h^2(h_t^2 + g_t^2) \geq 0$$

h_t and g_t are non-zero by regularity of α. So $|\phi_t \times \phi_\theta| = 0$ precisely when $h(t) = 0$, when the curve touches the axis of revolution. So a point $\phi(t,\theta) \in S$ is regular $\iff h(t) \neq 0$.

In general $g(t)$ measures the distance along the curve α whereas $h(t)$ measures the distance from the axis of revolution. So if instead $\alpha(t) = (0, g(t), h(t))$ is a curve in the yz-plane, rotating it around the z-axis gives the parametrisation $\phi(t,\theta) = (h(t)\sin\theta, g(t), h(t)\cos\theta)$.

Lemma 3.6. The parametrisation of a surface of revolution has orthogonal velocity vectors.

Proof. Suppose $\phi(t,\theta) = (h(t), g(t)\cos\theta, g(t)\sin\theta)$. Then

$$\begin{aligned}\phi_t &= (h'(t), g'(t)\cos\theta, g'(t)\sin\theta)\\ \phi_\theta &= (0, -g(t)\sin\theta, g(t)\cos\theta)\\ \implies \phi_t \cdot \phi_\theta &= -g(t)g'(t)\sin\theta\cos\theta + g(t)g'(t)\sin\theta\cos\theta = 0\end{aligned}$$

□

3.2 Curvature

We introduce two fundamental objects: the first and second fundamental forms. The definitions of these operators and the Weingarten map are from [HD10] whereas the derivation of their coefficients is from [Opr00].

Definition 3.7. The **first fundamental form** $I(v,w)$ for some $v, w \in T_pS$ is the symmetric operator

$$I(v,w) = v \cdot w$$

and its quadratic form is then given by

$$I(v) = I(v,v) = v \cdot v = |v|^2$$

The first fundamental form describes how the surface ϕ distorts lengths from their usual measurements in $\mathbb{R}^3$. Suppose γ is a unit speed curve with tangent vector γ', then $\gamma' \cdot \gamma' = |\gamma'|^2 = 1$ and so

$$\begin{aligned}1 &= \gamma' \cdot \gamma'\\ &= (u'\phi_u + v'\phi_v) \cdot (u'\phi_u + v'\phi_v)\\ &= (\phi_u \cdot \phi_u)u'^2 + (\phi_v \cdot \phi_u + \phi_u \cdot \phi_v)u'v' + (\phi_v \cdot \phi_v)v'^2\end{aligned}$$

$$
\begin{aligned}
&= I(\phi_u)u'^2 + 2I(\phi_u, \phi_v)u'v' + I(\phi_v)v'^2 \\
&= Eu'^2 + 2Fu'v' + Gv'^2
\end{aligned}
$$

where

$$
\begin{aligned}
E &= I(\phi_u) = \phi_u \cdot \phi_u = |\phi_u|^2 \\
F &= I(\phi_u, \phi_v) = \phi_u \cdot \phi_v \\
G &= I(\phi_v) = \phi_v \cdot \phi_v = |\phi_v|^2
\end{aligned}
$$

are called the **coefficients of the metric** or the **coefficients of** I.

Definition 3.8. Choose some point $p \in D$ and a surface parametrisation $\phi : D \to S$. Then the **Weingarten map** $\Omega_p : T_pS \to T_pS$ is the self-adjoint linear mapping

$$\Omega_p(v_1\phi_u(p) + v_2\phi_v(p)) = -v_1 N_u(p) - v_2 N_v(p)$$

Remark 3.9. Self-adjoint means that $\forall\, v, w \in T_pS$, $\Omega_p(v) \cdot w = v \cdot \Omega(w)$.

Remark 3.10. The codomain of Ω_p is indeed T_pS:

$$
\begin{aligned}
N \cdot N = 1 &\implies (N \cdot N_u = 0) \wedge (N \cdot N_v = 0) \\
&\implies N_u(p), N_v(p) \in T_pS \\
&\implies \Omega_p(v) \in T_pS
\end{aligned}
$$

Definition 3.11. The **second fundamental form** $II(v, w)$ for some $v, w \in T_pS$ is the symmetric operator

$$II(v, w) = \Omega_p(v) \cdot w$$

and its quadratic form is

$$II(v) = II(v, v) = \Omega_p(v) \cdot v = -v \cdot N'$$

We also introduce a formulation of a curve on the surface (from [HD10]).

Suppose we have a curve α in $D \subseteq \mathbb{R}^2$ starting at p:

$$
\begin{aligned}
\alpha : [0, \epsilon] &\to D \\
0 &\mapsto p \\
t &\mapsto (\alpha^1(t), \alpha^2(t))
\end{aligned}
$$

Then the projection of α onto a surface S under a parametrisation $\phi : D \to S$ gives a curve γ on the surface.

Definition 3.12. The **surface curve** $\gamma : [0, \epsilon] \to S$ is attained by projecting α under the map ϕ:

$$\begin{aligned}\gamma &= \phi \circ \alpha \\ 0 &\mapsto \phi(p) = w \\ t &\mapsto \phi(\alpha(t))\end{aligned}$$

With initial velocity $\gamma'(0) = \phi'(p)\alpha'(0) \in T_p\phi$.

Assume temporarily that γ is a unit speed curve so that $|\gamma'(s)| = 1$. So then the unit tangent vector of γ is $T(s) = \gamma'(s)$. Then the curvature κ of γ is $\kappa(s) = |T'(s)|$. The surface normal along $\gamma(s)$ is given by $N(\gamma(s))$.

Definition 3.13. The **side normal** B is

$$B = N \times T$$

Definition 3.14. The **moving orthonormal frame** [HD10] is the collection of unit vectors

$$\{T(t), B(t), N(t)\}$$

along the curve $\gamma(t)$ for $0 \leq t \leq \epsilon$, where $T_tS = span\{T(t), B(t)\}$ and $N(t)$ is orthogonal to this plane.

Lemma 3.15. Since $T \cdot T = 1$, we have that $T \cdot T' = 0$ and so dT/ds is orthogonal to the tangent vector and so lies in the span of N and B. That is,

$$\frac{dT}{ds} = \kappa_n N + \kappa_g B$$

The following definitions of the components of curvature are from [HD10].

Definition 3.16. The **normal curvature** κ_n is the component of the curvature κ of the surface curve γ acting in the normal direction:

$$\kappa_n = \frac{dT}{ds} \cdot N$$

It measures the acceleration of γ in the normal direction.

Definition 3.17. The **geodesic curvature** κ_g is the component of the curvature κ of the surface curve γ acting in the side normal direction:

$$\kappa_g = \frac{dT}{ds} \cdot B$$

It measures the acceleration of γ in the side normal direction.

Remark 3.18. If $\kappa_g(t) = 0$ at all points $0 \leq t \leq \epsilon$, then γ is a geodesic, i.e. it is a straight line.

Lemma 3.19. The curvature κ can now be expressed as

$$\kappa = \sqrt{\kappa_n^2 + \kappa_g^2}$$

This leads us to the following lemma, which is derived from a combination of discussion in [Opr00] and [HD10].

Lemma 3.20. The normal curvature of a unit-speed curve $\gamma(s)$ is $\kappa_n = -T \cdot N' = II(T)$, where $T = d\gamma/ds$.

Proof. We know that T is a tangent vector and N is orthogonal to the tangent plane, so we have that

$$\begin{aligned}
&T \cdot N = 0 \\
\implies &(T \cdot N)' = 0 \\
\implies &(T' \cdot N) + (T \cdot N') = 0 \\
\implies &T' \cdot N = -T \cdot N' \\
\implies &\kappa_n = -T \cdot N'
\end{aligned}$$

where $N' = u'N_u + v'N_v$, $u' = du/ds$ and $v' = dv/ds$. Then

$$\begin{aligned}
\kappa_n &= -T \cdot N' \\
&= -(u'\phi_u + v'\phi_v) \cdot (u'N_u + v'N_v) = II(T) \\
&= (-\phi_u \cdot N_u)u'^2 - (\phi_v \cdot N_u + \phi_u \cdot N_v)u'v' - (\phi_v \cdot N_v)v'^2
\end{aligned}$$

□

Remark 3.21. The coefficients of the second fundamental form l, $2m$ and n are

$$\begin{aligned}
l &= -\phi_u \cdot N_u \\
2m &= -(\phi_v \cdot N_u + \phi_u \cdot N_v) \\
n &= -\phi_v \cdot N_v \\
\implies \kappa_n &= lu'^2 + 2mu'v' + nv'^2
\end{aligned}$$

So we have shown that the normal curvature κ_n of a unit-speed curve $\gamma(s)$ at $s = s_0$ in the direction $\gamma'(s_0)$ is given by $II(\gamma'(s_0))$, where the derivatives are taken in terms of arc length.

But what about an arbitrary parametrisation? If $\gamma(t)$ is not unit-speed then $ds/dt = |\gamma'| = \sqrt{I(\gamma')}$ (definition 2.6) and so at $\gamma(0)$ in the direction $\gamma'(0)$

$$\frac{d\gamma}{dt} = \frac{d\gamma}{ds}\frac{ds}{dt} = \frac{d\gamma}{ds}\sqrt{I\Big(\frac{d\gamma}{dt}\Big)} = \Big|\frac{d\gamma}{dt}\Big|\frac{d\gamma}{ds}$$

$$\implies \frac{d\gamma}{ds} = \frac{1}{|d\gamma/dt|}\frac{d\gamma}{dt}$$
$$\implies \kappa_n = II\Big(\sqrt{I(\gamma'(0))}\gamma'(0)\Big) = \frac{II(\gamma'(0))}{I(\gamma'(0))}$$

And so we have shown that the normal curvature of a curve $\gamma(t)$ with velocity $\gamma'(t)$ at $t = t_0$ is

$$\kappa_n = \frac{II(\gamma'(t_0))}{I(\gamma'(t_0))}$$

For further discussion on this derivation, see [HD10].

Lemma 3.22. The coefficients of II may be computed using

$$\begin{aligned} l &= \phi_{uu} \cdot N \\ m &= \phi_{uv} \cdot N \\ n &= \phi_{vv} \cdot N \end{aligned}$$

Proof. (From [Opr00]). We have that ϕ_u and ϕ_v are tangent vectors. So both are orthogonal to N, i.e. $\phi_u \cdot N = 0$ and $\phi_v \cdot N = 0$. Therefore

$$\begin{aligned} &\frac{d}{du}(\phi_u \cdot N) = 0 \\ &\frac{d}{du}(\phi_v \cdot N) = 0 \\ &\frac{d}{dv}(\phi_u \cdot N) = 0 \\ &\frac{d}{dv}(\phi_v \cdot N) = 0 \\ \implies &\phi_{uu} \cdot N = -\phi_u \cdot N_u = l \\ &\phi_{uv} \cdot N = -\phi_v \cdot N_u \\ &\phi_{vu} \cdot N = -\phi_u \cdot N_v \\ &\phi_{vv} \cdot N = -\phi_v \cdot N_v = n \\ \implies &\phi_{uv} + \phi_{vu} = -(\phi_v \cdot N_u + \phi_u \cdot N_v) = 2m \\ \implies &m = \phi_{uv} \cdot N \end{aligned}$$

□

Example 3.23. Consider the cone described by the parametrisation

$$\phi(t, \theta) = (t, \frac{t}{2}\cos\theta, \frac{t}{2}\sin\theta)$$

The partial derivatives are

$$\phi_t = (1, \frac{1}{2}\cos\theta, \frac{1}{2}\sin\theta)$$
$$\phi_\theta = (0, -\frac{t}{2}\sin\theta, \frac{t}{2}\cos\theta)$$

And the surface normal is

$$\phi_t \times \phi_\theta = \Big(\frac{t}{4}, -\frac{t}{2}\cos\theta, -\frac{t}{2}\sin\theta\Big) \implies |\phi_t \times \phi_\theta| = \frac{t\sqrt{5}}{4}$$
$$\implies N = \Big(\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\cos\theta, -\frac{2}{\sqrt{5}}\sin\theta\Big)$$

This is pointing towards inwards towards the x-axis. Consider the curve $\alpha(\theta) = (1/2, \theta)$ so that the projected curve $\gamma = \phi(\alpha)$ traces out a circle on the cone at $t = 1/2$.

$$\gamma(\theta) = (\frac{1}{2}, \frac{1}{4}\cos\theta, \frac{1}{4}\sin\theta)$$
$$\gamma'(\theta) = (0, -\frac{1}{4}\sin\theta, \frac{1}{4}\cos\theta) \implies |\gamma'(\theta)| = \frac{1}{4}$$
$$\implies T = (0, -\sin\theta, \cos\theta)$$
$$\implies \frac{dT}{ds} = \frac{\gamma''(\theta)}{|\gamma'(\theta)|} = (0, -4\cos\theta, -4\sin\theta) \implies \kappa = 4$$

The side normal vector is

$$B = N \times T = \Big(-\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\cos\theta, -\frac{1}{\sqrt{5}}\sin\theta\Big)$$

So the components of curvature are

$$\kappa_n = \frac{dT}{ds} \cdot N = \frac{8}{\sqrt{5}}$$
$$\kappa_g = \frac{dT}{ds} \cdot B = \frac{4}{\sqrt{5}}$$
$$\implies \kappa = \sqrt{\kappa_n^2 + \kappa_g^2} = 4 = \Big|\frac{dT}{ds}\Big|$$

$\kappa_g \neq 0 \implies \gamma$ is not a geodesic.

The derivatives of the unit surface normal are

$$N_t = (0, 0, 0)$$

$$N_\theta = \Big(0, \frac{2}{\sqrt{5}}\sin\theta, -\frac{2}{\sqrt{5}}\cos\theta\Big)$$

Take the unit speed tangent vector $w = (0, 0, 1)$ on the surface of the cone at the point $\phi(p)$ for $p = (1/2, 0)$.

$$\begin{aligned}
w &= w_1\phi_t(p) + w_2\phi_\theta(p) = 4\phi_\theta(p) \\
&= (0, 0, 1) \\
\Longrightarrow \quad \Omega(w) &= -4N_\theta = -4\Big(0, 0, -\frac{2\sqrt{5}}{5}\Big) \\
&= \Big(0, 0, \frac{8\sqrt{5}}{5}\Big) \\
\Longrightarrow \quad II(w) &= \Omega(w)\cdot w \\
&= \frac{8\sqrt{5}}{5} \\
&= \kappa_n
\end{aligned}$$

This agrees with our earlier calculation. Now suppose instead of a unit speed tangent vector we have $w = \gamma'(0) = (0, 0, 1/4)$. Then the second fundamental form is instead

$$\begin{aligned}
w &= \phi_\theta(p) \\
\Longrightarrow \quad \Omega(w) &= \Big(0, 0, \frac{2}{\sqrt{5}}\Big) \\
\Longrightarrow \quad II(w) &= \frac{\sqrt{5}}{10} \neq \kappa_n
\end{aligned}$$

The first fundamental form at w is $I(w) = 1/16$ and so

$$\frac{II(w)}{I(w)} = 16\frac{\sqrt{5}}{10} = \frac{8\sqrt{5}}{5} = \kappa_n$$

as expected. The normal curvature may also be computed using the formula $\kappa_n = -T \cdot N'$:

$$\begin{aligned}
N' &= \frac{dN}{ds} = \frac{dN}{dt}\frac{dt}{ds} + \frac{dN}{d\theta}\frac{d\theta}{ds} \\
&= \frac{1}{(t/2)}\Big(0, \frac{2}{\sqrt{5}}\sin\theta, -\frac{2}{\sqrt{5}}\cos\theta\Big) \\
&= (0, \frac{4}{t\sqrt{5}}\sin\theta, -\frac{4}{t\sqrt{5}}\cos\theta) \\
\Longrightarrow \quad N'(p) &= \Big(0, 0, -\frac{8\sqrt{5}}{5}\Big)
\end{aligned}$$

$$\implies -T \cdot N'(p) = (0,0,-1) \cdot N'(p)$$
$$= \frac{8\sqrt{5}}{5} = \kappa_n$$

3.2.1 Mean Curvature

Definition 3.24. The **mean curvature** H is defined by the relation

$$2H = \kappa_1 + \kappa_2$$

where κ_1, κ_2 are the normal curvatures associated to any two perpendicular tangent vectors.

It turns out [Opr00] that

$$H = \frac{En + Gl - 2Fm}{2(EG - F^2)} \tag{4}$$

which depends only on the coefficients of the first and second fundamental forms and not on the particular tangent vectors chosen.

Definition 3.25. A surface is **minimal** if $H = 0$.

This condition is saying that on a minimal surface, any two perpendicular tangent directions have equal and opposite normal curvature. So either the surface is a plane in which case $\kappa_1 = \kappa_2 = 0$, or $\kappa_1 = -\kappa_2$.

Remark 3.26. It's sufficient to verify that $En + Gl - 2Fm = 0$ to see if a surface is minimal.

3.3 Catenoid

A catenary is the curve traced out by a freely hanging chain suspended between two points. The shape is that which minimises its own total gravitational potential energy [Opr00].

It is described by the equation

$$y = a \cosh \frac{x}{a}$$

where $(0, a)$ is the point on the catenary that is the closest to the x-axis.

A catenoid is a wormhole-like surface created by rotating a catenary around its gravitational axis. Some examples are shown in fig. 3. It is parametrised by

$$\phi(t,\theta) = (t, (a \cosh \frac{t}{a}) \cos\theta, (a \cosh \frac{t}{a}) \sin\theta)$$

Since $\cosh(t) \neq 0$ for real t, ϕ is a regular surface $\iff a \neq 0$.

The partial derivatives are

$$\phi_t = (1, \sinh\frac{t}{a}\cos\theta, \sinh\frac{t}{a}\sin\theta)$$
$$\phi_\theta = (0, -a\cosh\frac{t}{a}\sin\theta, a\cosh\frac{t}{a}\cos\theta)$$

and the surface normal is given by

$$\begin{aligned}\phi_t \times \phi_\theta &= \Big(a\sinh\frac{t}{a}\cosh\frac{t}{a}, -a\cos\theta\cosh\frac{t}{a}, -a\sin\theta\cosh\frac{t}{a}\Big)\\ \implies |\phi_t \times \phi_\theta| &= a\cosh^2\frac{t}{a}\\ \implies N &= (\tanh\frac{t}{a}, -\cos\theta\,\mathrm{sech}\,\frac{t}{a}, -\sin\theta\,\mathrm{sech}\,\frac{t}{a})\end{aligned}$$

3.3.1 Minimality

The catenoid is one of the most famous minimal surfaces. To verify its minimality we will compute the coefficients of I and II. The coefficients of the metric, E, F and G are

$$E = I(\phi_t) = \cosh^2\frac{t}{a}$$
$$F = I(\phi_t, \phi_\theta) = 0$$
$$G = I(\phi_\theta) = a^2\cosh^2\frac{t}{a}$$

and the second partial derivatives of ϕ are

$$\phi_{tt} = \Big(0, \frac{1}{a}\cosh\frac{t}{a}\cos\theta, \frac{1}{a}\cosh\frac{t}{a}\sin\theta\Big)$$
$$\phi_{t\theta} = \Big(0, -\sin\theta\sinh\frac{t}{a}, \cos\theta\sinh\frac{t}{a}\Big)$$
$$\phi_{\theta\theta} = \Big(0, -a\cosh\frac{t}{a}\cos\theta, -a\cosh\frac{t}{a}\sin\theta\Big)$$

so the coefficients of the second fundamental form, l, m and n are

$$l = \phi_{tt} \cdot N = -\frac{1}{a}$$
$$m = \phi_{t\theta} \cdot N = 0$$
$$n = \phi_{\theta\theta} \cdot N = a$$

and so we can see that $H = 0$ since

$$En + Gl - 2Fm = a\cosh^2\frac{t}{a} - a\cosh^2\frac{t}{a} = 0$$

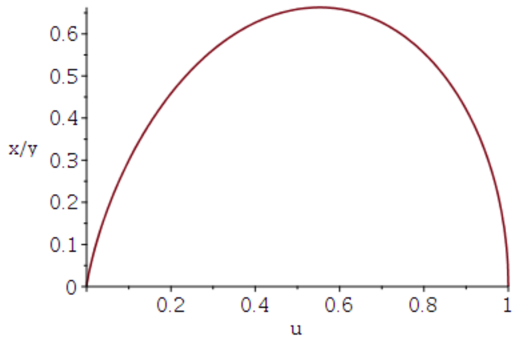


Figure 2: Behaviour of x/y as u is allowed to vary.

3.3.2 Possibility Space

This section is an analysis of some special points on a catenary and how they relate to the properties of their associate catenoid. Much of the source material is from [Opr00].

Let $u = a/y$. The equation of a catenary can then be expressed as

$$\frac{x}{y} = u \cosh^{-1} \frac{1}{u}$$

Figure 2 shows how the ratio between x and y changes as u is allowed to vary. We can see that for each value of x/y—except for some maximum—there are two values of u that produce a catenary passing through $(-x, y)$ and (x, y).

To find the maximum of the curve we solve the equation

$$\frac{d}{du}(u \cosh^{-1} \frac{1}{u}) = 0$$

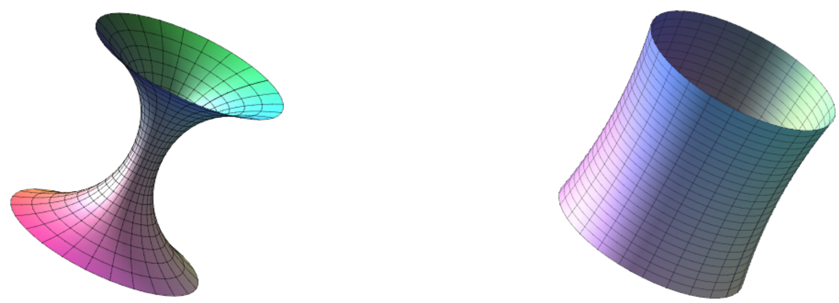
and see that the critical point is at $u \approx 0.5524341245$ for $u \in [0, 1]$. So then the maximum value of $x/y \approx 0.6627434192$.

Suppose we pick a point (x_0, y_0) and ask if we can produce a catenary that passes through this point. What we have shown is that this is possible precisely when $x_0/y_0 \lessapprox 0.6627434192$.

Let $x_0/y_0 = 0.4/1$. Then the two possible catenaries are given by the parameters

$$u_0 = a_0 \approx 0.1579623972$$
$$u_1 = a_1 \approx 0.9107379943$$

The catenoids C_0 and C_1 associated with these values are shown in fig. 3.



(a) Narrow catenoid C_0 produced using the catenary $a_0 \cosh x/a_0$.

(b) Thick catenoid C_1 produced using the catenary $a_1 \cosh x/a_1$.

Figure 3: Catenoids created by revolving two different catenaries around their gravitational axis.

We will now show that the narrow catenoids with $u \lessapprox 0.5524341245$ always have a larger surface area than their thick counterparts.

Definition 3.27. The **surface area** [HD10] of a surface parametrised by $\phi : D \to S$ is

$$A(\phi) = \int_D |\phi_u \times \phi_v| \, du \, dv$$

Consider a catenary in the xy-plane connecting the points $(-x_0, y_0)$ and (x_0, y_0). Rotating this catenary forms a catenoid connecting the two disks centered at $(-x_0, 0)$ and $(x_0, 0)$ with radius y_0. The surface area of this catenoid is given by

$$A(\phi) = 2\pi \int_{-x_0}^{x_0} a \cosh^2 \frac{x}{a} \, dx$$

Consider the ratio between the area of the surface and the area of the two disks that it connects:

$$\frac{A(\phi)}{2\pi y_0^2} = \frac{a}{y_0^2} \int_{-x_0}^{x_0} \cosh^2 \frac{x}{a} \, dx$$

For C_0 this quantity is ≈ 1.050630087 whereas for C_1 it is ≈ 0.7772798291. Since both connect the same two disks, this means the thicker catenoid C_1 has the smaller surface area. More generally,

$$\frac{A(\phi)}{2\pi y_0^2} = \frac{a}{y_0^2} \int_{-x_0}^{x_0} \cosh^2 \frac{x}{a} \, dx$$

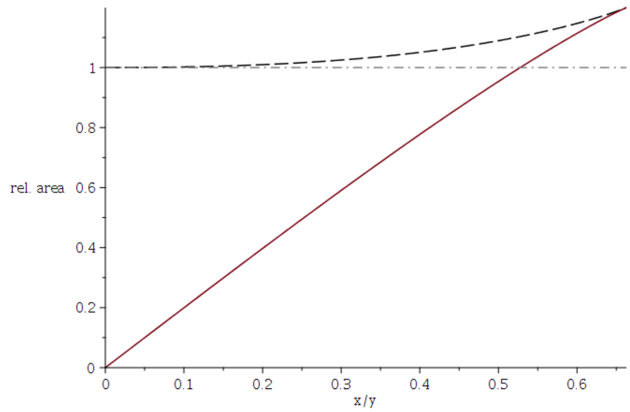


Figure 4: Relative area of catenoid for some choice of connected disks. Dashed line is narrow catenoids with values of $u \lessapprox 0.5524341245$ whereas solid line is thicker catenoids with values of $u \gtrapprox 0.5524341245$. Horizontal line is the area of the two disks.

$$
\begin{aligned}
&= \frac{a}{2y_0^2}\left(a \sinh \frac{2x_0}{a} + 2x_0\right) \\
&= \frac{a}{y_0}\left(\frac{a}{y_0} \sinh \frac{x_0}{a} \cosh \frac{x_0}{a} + \frac{x_0}{y_0}\right) \\
&= u\left(u \sinh \frac{x_0}{a} \cosh \frac{x_0}{a} + u \cosh^{-1} \frac{1}{u}\right) \\
&= u^2\left(\frac{1}{u} \sinh \frac{x_0}{a} + \cosh^{-1} \frac{1}{u}\right) \\
&= u^2\left(\frac{1}{u} \sqrt{\cosh^2 \frac{x_0}{a} - 1} + \cosh^{-1} \frac{1}{u}\right) \\
&= u^2\left(\frac{1}{u} \sqrt{\frac{1}{u^2} - 1} + \cosh^{-1} \frac{1}{u}\right)
\end{aligned}
$$

This is a formulation of the ratio in terms of u. Now for each value of x_0/y_0 we can plot the relative surface area of the associated catenoids (fig. 4).

The graph shows that the narrower catenoids have strictly larger surface area than their thicker counterparts. In fact, their surface area is greater than that of the disks that they connect. On the other hand, the thicker catenoids have area smaller than that of the disks that they connect, up to a certain point.

The point where the narrow and thick catenoids have the same surface area is $u = 0.5524341245$—the critical point of fig. 2. At this point there is only a single possible catenoid connecting the two disks and it has maximal surface area.

What is the value of x/y for which the surface area of the catenoid is equal to the two disks that it connects? Consider the surface area of the catenoid by itself:

$$
\begin{aligned}
A(\phi) &= 2\pi \int_{-x_0}^{x_0} a \cosh^2 \frac{x}{a}\, dx \\
&= 2\pi a^2 \sinh \frac{x_0}{a} \cosh \frac{x_0}{a} + 2\pi a x_0
\end{aligned}
$$

Set $y_0 = 1$ and let $x_0 \in [0, 0.6627434192]$. This is a range over every possible x/y ratio where a catenoid may be constructed between the disks. Solving $A(\phi) - 2\pi = 0$ yields $x_s/y_s = 0.5276973970$. This is the so-called Goldschmidt discontinuous solution where if $x_0/y_0 \geq x_s/y_s$ then two disks give an absolute minimum for surface area. For more discussion on this see [Opr00].

3.3.3 Helicoid

The helicoid (fig. 5a) is a minimal surface closely related to the catenoid. It can be generated by the parametrisation

$$\phi(u, v) = (v \cos u, v \sin u, u)$$

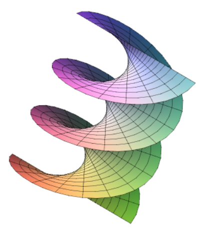
This describes a rotating rising line, somewhat analogous to the rotating rising point that generates a helix. Surfaces like this are called ruled surfaces.

$$
\begin{aligned}
E &= v^2 + 1 \\
F &= 0 \\
G &= 1 \\
l &= 0 \\
m &= \frac{1}{\sqrt{v^2 + 1}} \\
n &= 0 \\
\implies H &= En + Gl - 2Fm = 0
\end{aligned}
$$

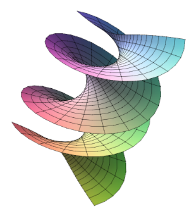
So the helicoid is indeed a minimal surface. There exists a continuous deformation [Opr00] between the helicoid and the catenoid given by the parametrisation

$$
\begin{aligned}
\xi(t) = (\, & \sin u \cos t \sinh v + \sin t \cos u \cosh v, \\
& - \cos t \sinh v \cos u + \sin t \cosh v \sin u, \\
& u \cos t + v \sin t)
\end{aligned}
$$

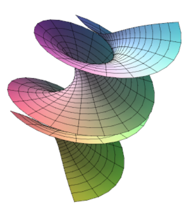
where $t = 0$ corresponds to a helicoid and $t = \pi/2$ is a catenoid. This deformation is shown in fig. 5.



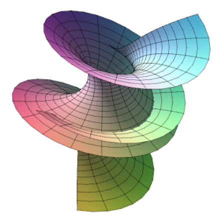
(a) $\xi(0)$



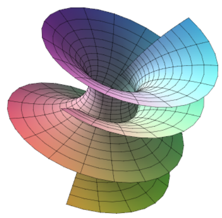
(b) $\xi(\pi/10)$



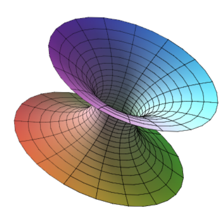
(c) $\xi(\pi/5)$



(d) $\xi(3\pi/10)$



(e) $\xi(2\pi/5)$



(f) $\xi(\pi/2)$

Figure 5: Plot of ξ showing the deformation of a helicoid into a catenoid.

4 Minimal Surfaces

Suppose M is a surface that can be locally described as a graph over a plane (Monge parametrisation, section 3.1.1). Let

$$\phi(u,v) = (u, v, f(u,v))$$

Definition 4.1 (The Minimal Surface Equation)**.** A surface is minimal if and only if it can be locally expressed as the graph of a solution to

$$(1 + f_v^2)f_{uu} - 2f_u f_v f_{uv} + (1 + f_u^2)f_{vv} = 0 \tag{5}$$

See [HD10] and [Opr00] for further discussion.

Since any solution of eq. (5) is a minimal surface, we can "discover" minimal surfaces by finding solutions. Suppose we add some arbitrary constraint that makes the differential equation solvable.

Example 4.2. We want to solve eq. (5) for some $f(u,v)$ so that $\phi(u,v)$ is minimal. So suppose f is separable so that the value of $f(u,v)$ is separately dependent on the inputs x and y. More precisely,

$$\begin{aligned} f(u,v) &= g(u) + h(v) \\ \implies f_u &= g'(u) \\ f_v &= h'(v) \\ f_{uu} &= g''(u) \\ f_{vv} &= h''(v) \\ f_{uv} &= 0 \end{aligned}$$

Substituting these into the minimal surface equation yields

$$\begin{aligned} (1 + h'(v)^2)g''(u) + (1 + g'(u)^2)h''(v) &= 0 \\ (1 + h'(v)^2)g''(u) &= -(1 + g'(u)^2)h''(v) \\ \frac{(1 + h'(v)^2)}{h''(v)} &= -\frac{(1 + g'(u)^2)}{g''(u)} \end{aligned}$$

But then the left side is dependent on only v whereas the right side is dependent on only u, and yet they are equal for any value of u or v. Another way of saying this is that any change in u has no effect on either side since v has not changed. Therefore both sides must be equal to the same constant c.

So consider the left side:

$$1 + h'(v)^2 = ch''(v)$$

Let $y = h'$. Then $dy = h''\,dv$ and so

$$\begin{aligned} 1 + y^2 &= cy' \\ 1 &= \frac{cy'}{1+y^2} \\ \int 1\,dv &= \int \frac{cy'}{1+y^2}\,dv \\ v &= \int \frac{c}{1+y^2} h''(v)\,dv \\ v &= c\int \frac{1}{1+y^2}\,dy \end{aligned}$$

Suppose $y = \tan x$. Then

$$\begin{aligned} \frac{d}{dy} y &= \frac{d}{dy}\tan x \\ 1 &= \sec^2 x\,\frac{dx}{dy} \\ \frac{dy}{dx} &= \sec^2 x = 1 + \tan^2 x \implies \frac{dx}{dy} = \frac{1}{1+y^2} \\ x &= \int \frac{1}{1+y^2}\,dy = \tan^{-1} y \\ \implies v &= c\int \frac{1}{1+y^2}\,dy = c\tan^{-1} y \end{aligned}$$

Where the constant of integration has been suppressed. So now we have that $v = c\tan^{-1}(y)$ so $y = \tan(v/c)$. Therefore,

$$\begin{aligned} \frac{dh}{dv} &= \tan\frac{v}{c} \\ h(v) &= \int \tan\frac{v}{c}\,dv \\ &= -c\ln\cos\frac{v}{c} \end{aligned}$$

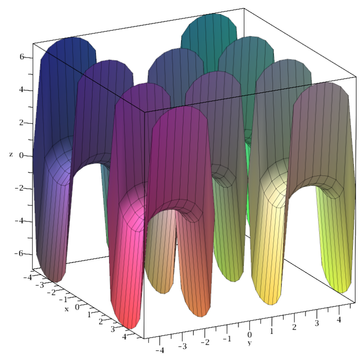
where the constant of integration was again suppressed. The calculation for $g(u)$ is identical except the sign is flipped so

$$\begin{aligned} f(u,v) &= c\ln\cos\frac{u}{c} - c\ln\cos\frac{v}{c} \\ &= c\Big(\ln\cos\frac{u}{c} - \ln\cos\frac{v}{c}\Big) \\ &= c\ln\frac{\cos(u/c)}{\cos(v/c)} \end{aligned}$$

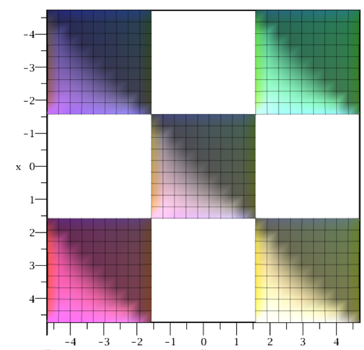
Therefore the parametrisation ϕ is given by

$$\phi(u,v) = \left(u, v, c \ln \frac{\cos(u/c)}{\cos(v/c)}\right)$$

This is *Scherk's first surface* (fig. 6). Notice that it is only defined on a checkerboard of the uv-plane as the logarithm is not defined for negative inputs.



(a) Side view showing the terrain.



(b) Checkerboard pattern can be seen from above.

Figure 6: Scherk's first surface for $c = 1$.

4.1 Soap Films

In this section we will introduce the relationship between minimal surfaces in differential geometry and soap films in real life. The derivation of the fundamental result of this section is from [Opr00].

The shape of a soap film is dictated by is surface tension σ. Surface tension acts to pull the film as tight as possible until some equilibrium is reached.

Definition 4.3. The **surface tension** σ is defined as the force per unit distance.

Definition 4.4. The **pressure** p is force per unit area.

In the context of soap films p is the pressure difference across the film. In other words it is the difference in pressure on the exterior and interior of the film. When p is zero, the film is in equilibrium with its environment.

Theorem 4.5 (First Principle of Soap Films)**.** A soap film minimises its surface area. [Opr00]

To see how this works consider a section of soap film that is expanded outwards due to an applied pressure p. More precisely, take a piece of film given by two perpendicular directions x and y on the surface and compute the work done to expand the surface area under some pressure.

Consider the x direction first. The original film is a sector of a circle with angle θ and radius R_1. The circumference of this sector is $x = (\theta/2\pi)2\pi R_1 = \theta R_1$. When this is expanded by an applied pressure p, the new radius is $R_1 + \delta r$ and the new circumference is $x + \delta x$.

$$\delta x = \theta \delta r \implies \theta = \frac{\delta x}{\delta r}$$

So then for the expanded section we have the relation

$$\frac{x + \delta x}{R_1 + \delta r} = \theta = \frac{x}{R_1}$$

and so we can write the new circumference of the expanded sector as

$$x + \delta x = \frac{x}{R_1}(R_1 + \delta r) = x\Big(1 + \frac{\delta r}{R_1}\Big)$$

Similarly for the y component with radius R_2 we have

$$y + \delta y = y\Big(1 + \frac{\delta r}{R_2}\Big)$$

Definition 4.6. The **work done** $W = FD$ where F is force and D is the distance over which the force acts. Equivalently $W = \sigma \delta S$ where δS is the change in surface area.

The work done to expand some section of soap film is

$$\begin{aligned} W &= FD \\ &= (pS)\delta r \\ &= (pxy)\delta r \\ &= \sigma \delta S \end{aligned}$$

and the change in surface area is given by

$$\begin{aligned} \delta S &= (x + \delta x)(y + \delta y) - xy \\ &= x\Big(1 + \frac{\delta r}{R_1}\Big) y \Big(1 + \frac{\delta r}{R_2}\Big) - xy \\ &= xy\delta r\Big(\frac{1}{R_1} + \frac{1}{R_2}\Big) + xy\frac{(\delta r)^2}{R_1 R_2} \end{aligned}$$

If δr is small, the higher order term can be ignored. So we have that

$$\begin{aligned} pxy\delta r &= \sigma\delta S \\ &= \sigma xy\delta r\Big(\frac{1}{R_1} + \frac{1}{R_2}\Big) \\ \implies p &= \sigma\Big(\frac{1}{R_1} + \frac{1}{R_2}\Big) \end{aligned}$$

Hence we arrive at the following fundamental result:

Theorem 4.7. The **Laplace-Young equation** is given by

$$p = \sigma\Big(\frac{1}{R_1} + \frac{1}{R_2}\Big)$$

Combining our earlier requirement that the two directions x and y are perpendicular with the fact that the normal curvature of a circle segment with radius r is $1/r$ we get

$$2H = \Big(\frac{1}{R_1} + \frac{1}{R_2}\Big)$$

and so the Laplace-Young equation may be rewritten as

$$p = 2\sigma H$$

where H is the mean curvature of the film. This leads to the observation:

Corollary 4.8. A soap film in equilibrium is a minimal surface.

Proof. When the soap film is in equilibrium with its environment,

$$p = 0 \implies H = 0$$

and so the film is a minimal surface. □

4.2 Isothermal Coordinates

In this section we will introduce a special type of parametrisation that simplifies a lot of what we have already discussed, and sets the foundations for what comes later. The source material is from [Opr00].

Definition 4.9. A parametrisation ϕ is **isothermal** if $E = \phi_u \cdot \phi_u = \phi_v \cdot \phi_v = G$ and $F = \phi_u \cdot \phi_v = 0$.

Remark 4.10. If ϕ is isothermal, then the definition of mean curvature reduces to

$$H = \frac{n + l}{2E}$$

Remark 4.11. If ϕ is isothermal then ϕ is minimal $\iff n + l = \phi_{uu} \cdot N + \phi_{vv} \cdot N = 0$.

Theorem 4.12. Isothermal coordinates exist on any minimal surface.

Proof. See [Opr00]. □

Definition 4.13. The **laplace operator** Δ is defined by $\Delta\phi = \phi_{uu} + \phi_{vv}$.

Definition 4.14. A function f is **harmonic** if $\Delta f = 0$.

The following theorem is the main result of this section. Its proof is a simplified and more specific adaptation of a similar result in [Opr00].

Theorem 4.15. If M is a regular surface with isothermal coordinates ϕ, M is minimal $\iff$ ϕ is harmonic.

Proof. Since ϕ is isothermal we have

$$\begin{aligned} H &= \frac{n+l}{2E} \\ \implies 2EH &= n + l \\ &= \phi_{uu} \cdot N + \phi_{vv} \cdot N \\ &= N \cdot (\phi_{uu} + \phi_{vv}) \\ &= N \cdot \Delta\phi \end{aligned}$$

So suppose M is minimal. Then $N \cdot \Delta\phi = 0 \implies \Delta\phi = 0$ since $|N| = 1$. Hence ϕ is harmonic.

Conversely suppose ϕ is harmonic. So then $\Delta\phi = 0 \implies H = 0$ since $|N| = 1$ as before and $E \neq 0$. Therefore M is minimal. □

4.3 Complex Analysis

Complex analysis turns out to be very useful in generating minimal surfaces. In this section we define some well-known foundational results that will be useful later. For further reading see [ST83] and [Opr00].

Suppose f is a function $f(z) = r(u, v) + is(u, v)$ where $z = u + iv$ and r, s are real-valued functions of real numbers u and v.

Definition 4.16. The **Cauchy-Riemann equations** are the conditions

$$\frac{\partial r}{\partial u} = \frac{\partial s}{\partial v} \tag{6}$$

$$\frac{\partial s}{\partial u} = -\frac{\partial r}{\partial v} \tag{7}$$

Definition 4.17. A function f is **differentiable** at z if $\partial r/\partial u$, $\partial s/\partial u$, $\partial r/\partial v$ and $\partial s/\partial v$ all exist and f conforms to the Cauchy-Riemann equations at z.

Definition 4.18. A function is **holomorphic** if it is differentiable at all points in its domain.

Definition 4.19. A function is **meromorphic** if it is holomorphic over all of its domain except a few isolated points. These points are called its **singularities**.

Theorem 4.20. If $f(z) = r(u, v) + is(u, v)$ is holomorphic then both r and s are harmonic.

Proof. (From [Opr00]). Suppose f is holomorphic. So then by the Cauchy-Riemann equations,

$$\begin{aligned} \frac{\partial^2 r}{\partial u^2} &= \frac{\partial}{\partial u}\frac{\partial s}{\partial v} = \frac{\partial^2 s}{\partial u \partial v} \\ \frac{\partial^2 r}{\partial v^2} &= -\frac{\partial}{\partial v}\frac{\partial s}{\partial u} = -\frac{\partial^2 s}{\partial u \partial v} \\ \implies \frac{\partial^2 r}{\partial u^2} &+ \frac{\partial^2 r}{\partial v^2} = 0 \\ &\implies \Delta r = 0 \end{aligned}$$

so r is harmonic. Similarly for s we have

$$\begin{aligned} \frac{\partial^2 s}{\partial u^2} &= -\frac{\partial}{\partial u}\frac{\partial r}{\partial v} = -\frac{\partial^2 r}{\partial u \partial v} \\ \frac{\partial^2 s}{\partial v^2} &= \frac{\partial}{\partial v}\frac{\partial r}{\partial u} = \frac{\partial^2 r}{\partial u \partial v} \\ \implies \frac{\partial^2 s}{\partial u^2} &+ \frac{\partial^2 s}{\partial v^2} = 0 \\ &\implies \Delta s = 0 \end{aligned}$$

so s is also harmonic. □

Definition 4.21. Consider a parametrisation $\phi(u, v)$ with complex coordinates. If $z = u + iv$ and $\bar{z} = u - iv$ then we denote

$$\begin{aligned} \frac{\partial}{\partial z} &= \frac{1}{2}\Big(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\Big) \\ \frac{\partial}{\partial \bar{z}} &= \frac{1}{2}\Big(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\Big) \end{aligned}$$

Lemma 4.22. If $f(z) = r(u, v) + is(u, v)$ is a function then f is holomorphic $\iff$ $\partial f/\partial \bar{z} = 0$.

Proof. Suppose f is holomorphic. Then

$$\begin{aligned}\frac{\partial f}{\partial \bar{z}} &= \frac{1}{2}\Big(\frac{\partial f}{\partial u} + i\frac{\partial f}{\partial v}\Big) \\ &= \frac{1}{2}\Big(\frac{\partial r}{\partial u} + i\frac{\partial s}{\partial u} + i\Big(\frac{\partial r}{\partial v} + i\frac{\partial s}{\partial v}\Big)\Big) \\ &= \frac{1}{2}\Big(\frac{\partial r}{\partial u} + i\frac{\partial s}{\partial u} - i\frac{\partial s}{\partial u} - \frac{\partial r}{\partial u}\Big) = 0\end{aligned}$$

Conversely suppose that $\partial f/\partial \bar{z} = 0$. Then

$$\begin{aligned}\frac{\partial f}{\partial u} + i\frac{\partial f}{\partial v} &= 0 \\ \frac{\partial r}{\partial u} + i\frac{\partial s}{\partial u} + i\frac{\partial r}{\partial v} - \frac{\partial s}{\partial v} &= 0 \\ \implies \Big(\frac{\partial r}{\partial u} - \frac{\partial s}{\partial v}\Big) &= 0 \\ \Big(\frac{\partial s}{\partial u} + \frac{\partial r}{\partial v}\Big) &= 0 \\ \implies \Big(\frac{\partial r}{\partial u} = \frac{\partial s}{\partial v}\Big) \wedge \Big(\frac{\partial s}{\partial u} = -\frac{\partial r}{\partial v}\Big)&\end{aligned}$$

so f is holomorphic. ∎

Lemma 4.23. $\Delta f = f_{uu} + f_{vv} = 4(\partial/\partial z(\partial f/\partial \bar{z}))$.

Proof.

$$\begin{aligned}\frac{\partial}{\partial z}\Big(\frac{\partial f}{\partial \bar{z}}\Big) &= \frac{\partial}{\partial z}\Big(\frac{1}{2}(f_u + if_v)\Big) \\ &= \frac{1}{4}(f_{uu} + if_{uv} - if_{uv} + f_{vv}) \\ &= \frac{1}{4}(f_{uu} + f_{vv}) = \frac{1}{4}\Delta f \\ \implies \Delta f &= 4\Big(\frac{\partial}{\partial z}\Big(\frac{\partial f}{\partial \bar{z}}\Big)\Big)\end{aligned}$$

∎

Definition 4.24. Suppose $f = r + is$ is continuous and $\gamma(t) : [a, b] \to \mathbb{C}$ is a curve. Then the **complex integral** of f along γ is

$$\int_\gamma f = \int_a^b f(\gamma(t))\gamma'(t)\, dt$$

Theorem 4.25. The Fundamental Theorem of Calculus for complex integrals states that if f is holomorphic then

$$\int_\gamma f' = f(b) - f(a)$$

Theorem 4.26 (The Identity Theorem)**.** If f and g are two holomorphic functions on a connected, open region $D \in \mathbb{C}$, and $f(z_i) = g(z_i)$ for some convergent sequence $z_1, z_2, \ldots, z_n, \ldots \to \bar{z} \in D$, then $f = g$ on all of D.

This definition is from [Opr00]. For an alternate one see [ST83].

4.4 The Weierstrass-Enneper Representation

The Weierstrass-Enneper representation allows us to create minimal surfaces using holomorphic functions. The derivation here is from [Opr00].

Let M be a minimal surface wih isothermal parametrisation $\phi(u, v)$. Let $z = u+iv$ denote the corresponding complex coordinate. We have that $u = (z+\bar{z})/2$ and $v = -i(z - \bar{z})/2$ and so we can write

$$\phi(z, \bar{z}) = (\varphi^1(z, \bar{z}), \varphi^2(z, \bar{z}), \varphi^3(z, \bar{z}))$$

where φ^i are complex valued functions that happen to take real values.

As we defined in the previous section, we have that

$$\frac{\partial \varphi^i}{\partial z} = \frac{1}{2}\Big(\frac{\partial \varphi^i}{\partial u} - i\frac{\partial \varphi^i}{\partial v}\Big) = \frac{1}{2}(\varphi^i_u - i\varphi^i_v) = \varphi^i_z$$

So the first order derivative ϕ' is

$$\phi' = \frac{\partial \phi}{\partial z} = (\varphi^1_z, \varphi^2_z, \varphi^3_z)$$

We will use the following notation:

$$\begin{aligned}(\phi')^2 &= (\varphi^1_z)^2 + (\varphi^2_z)^2 + (\varphi^3_z)^2 \\ |\phi'|^2 &= |\varphi^1_z|^2 + |\varphi^2_z|^2 + |\varphi^3_z|^2\end{aligned}$$

where $|z| = \sqrt{u^2 + v^2}$ is the modulus of z. Consider the following:

$$\begin{aligned}(\varphi^i_z)^2 &= \Big(\frac{1}{2}(\varphi^i_u - i\varphi^i_v)\Big)^2 \\ &= \frac{1}{4}(\varphi^i_u - i\varphi^i_v)(\varphi^i_u - i\varphi^i_v) \\ &= \frac{1}{4}((\varphi^i_u)^2 - (\varphi^i_v)^2 - 2i\varphi^i_u\varphi^i_v)\end{aligned}$$

So then $(\phi')^2$ is

$$(\phi')^2 = (\varphi^1_z)^2 + (\varphi^2_z)^2 + (\varphi^3_z)^2$$

$$
\begin{aligned}
&= \frac{1}{4}\Big(\sum_{j=1}^{3}(\varphi_u^j)^2 - \sum_{j=1}^{3}(\varphi_v^j)^2 - 2i\sum_{j=1}^{3}\varphi_u^j\varphi_v^j\Big) \\
&= \frac{1}{4}(|\phi_u|^2 - |\phi_v|^2 - 2i(\phi_u \cdot \phi_v)) \\
&= \frac{1}{4}(E - G - 2iF)
\end{aligned}
$$

Lemma 4.27. ϕ is isothermal $\iff (\phi')^2 = 0$

Proof. Suppose ϕ is isothermal. Then

$$(\phi')^2 = \frac{1}{4}(E - G - 2iF) = \frac{1}{4}(E - E - 0) = 0$$

Conversely suppose $(\phi')^2 = 0$. So then

$$
\begin{aligned}
&\frac{1}{4}(E - G - 2iF) = \frac{E}{4} - \frac{G}{4} - \frac{iF}{2} = 0 + 0i \\
\implies &\Big(\frac{E}{4} - \frac{G}{4} = 0\Big) \wedge \Big(-\frac{F}{2} = 0\Big) \\
\implies &(E = G) \wedge (F = 0)
\end{aligned}
$$

and so ϕ is isothermal. $\square$

Now instead consider $|\phi'|^2$:

$$
\begin{aligned}
|\varphi_z^i|^2 &= \frac{1}{4}((\varphi_u^i)^2 + (\varphi_v^i)^2) \\
\implies |\phi'|^2 &= |\varphi_z^1|^2 + |\varphi_z^2|^2 + |\varphi_z^3|^2 \\
&= \frac{1}{4}\Big(\sum_{j=1}^{3}|\varphi_u^j|^2 + \sum_{j=1}^{3}|\varphi_v^j|^2\Big) \\
&= \frac{1}{4}(|\phi_u|^2 + |\phi_v|^2) = \frac{1}{4}(E + G)
\end{aligned}
$$

Lemma 4.28. If ϕ is an isothermal parametrisation of a regular surface, $|\phi'|^2 = E/2 \neq 0$.

Proof. If ϕ is isothermal, $E = G$ and so $|\phi'|^2 = 2E/4 = E/2$. This is non-zero by regularity. $\square$

Theorem 4.29. If M is a surface with isothermal parametrisation ϕ, M is minimal $\iff \phi'$ is holomorphic.

Proof. Suppose M is minimal. Therefore since ϕ is isothermal, we have that ϕ is harmonic (theorem 4.15) so $\Delta\phi = 0$. By lemma 4.23 then

$$4\Delta\phi = 0 = \frac{\partial}{\partial\bar{z}}\Big(\frac{\partial\phi}{\partial z}\Big) = \frac{\partial\phi'}{\partial\bar{z}}$$

so by lemma 4.22 ϕ' is holomorphic. But equality holds both ways so supposing that ϕ' is holomorphic implies that ϕ is harmonic. So ϕ isothermal $\implies M$ is minimal. □

Recall that an isothermal parametrisation can be found for any minimal surface M. This theorem is powerful because it says that M can be described near each of its points by a triple of holomorphic functions.

Suppose $\phi' = (\varphi^1_z, \varphi^2_z, \varphi^3_z)$ is a triple of holomorphic functions so that $(\phi')^2 = 0$. By the results above ϕ is an isothermal parametrisation for a minimal surface M. We can construct $\phi(z,\bar{z}) = (\varphi^1, \varphi^2, \varphi^3)$ explicitly by setting

$$\varphi^i(z,\bar{z}) = c_i + 2Re\Big\{\int \varphi^i_z \, dz\Big\}$$

To see how this works note that $dz = du + idv$ since $z = u + iv$. Also, $\varphi^i_z = 1/2(\varphi^i_u - i\varphi^i_v)$. Therefore

$$\begin{aligned}
\varphi^i_z dz &= \frac{1}{2}(\varphi^i_u - i\varphi^i_v)(du + idv) \\
&= \frac{1}{2}(\varphi^i_u du + \varphi^i_v dv + i(\varphi^i_u dv - \varphi^i_v du)) \\
\varphi^i_{\bar{z}} d\bar{z} &= \frac{1}{2}(\varphi^i_u + i\varphi^i_v)(du - idv) \\
&= \frac{1}{2}(\varphi^i_u du + \varphi^i_v dv - i(\varphi^i_u dv - \varphi^i_v du)) \\
\implies \varphi^i_z dz + \varphi^i_{\bar{z}} d\bar{z} &= \varphi^i_u du + \varphi^i_v dv = d\varphi^i
\end{aligned}$$

The last line is also equal to two times the real part of $\varphi^i_z dz$, so

$$d\varphi^i = 2Re\Big\{\varphi^i_z dz\Big\}$$

and so integrating both sides gives

$$\varphi^i = c_i + 2Re\Big\{\int \varphi^i_z \, dz\Big\}$$

So we can construct a minimal surface simply by selecting a triple of holomorphic functions $\vartheta = (\vartheta^1, \vartheta^2, \vartheta^3)$ with $(\vartheta)^2 = 0$. This setup implies the existence of an isothermal parametrisation ϕ for a minimal surface.

Suppose we have a holomorphic function f and a meromorphic function g so that fg^2 is holomorphic. Then set

$$\vartheta^1 = \frac{1}{2}f(1-g^2)$$
$$\vartheta^2 = \frac{i}{2}f(1+g^2)$$
$$\vartheta^3 = fg$$

This construction does satisfy our isothermal condition:

$$\begin{aligned}(\vartheta)^2 &= (\frac{1}{2}f(1-g^2))^2 + (\frac{i}{2}f(1+g^2))^2 + f^2g^2 \\ &= \frac{1}{4}f^2(-4g^2) + f^2g^2 \\ &= f^2g^2 - f^2g^2 \\ &= 0\end{aligned}$$

So now we arrive at the main result of this section: the Weierstrass-Enneper representation.

Theorem 4.30 (Weierstrass-Enneper I)**.** If f is holomorphic on a domain D, g meromorphic on D, and fg^2 holomorphic on D, then a minimal surface is defined by the parametrisation $\phi(z,\bar{z}) = (\varphi^1(z,\bar{z}), \varphi^2(z,\bar{z}), \varphi^3(z,\bar{z}))$ where

$$\varphi^1 = Re\Big\{\int f(1-g^2)\,dz\Big\}$$
$$\varphi^2 = Re\Big\{\int if(1+g^2)\,dz\Big\}$$
$$\varphi^3 = 2Re\Big\{\int fg\,dz\Big\}$$

The constant of integration $c_i = 0$.

Example 4.31. Let $f = -e^{-z}/2$ and $g = -e^z$. Then

$$\begin{aligned}\varphi^1 &= Re\Big\{\int f(1-g^2)\,dz\Big\} \\ &= Re\Big\{-\frac{1}{2}\int e^{-z} - e^z\,dz\Big\} \\ &= Re\Big\{\frac{e^z+e^{-z}}{2}\Big\} \\ &= Re\{\cosh z\} \\ &= Re\{\cosh u\cos v + i\sinh u\sin v\} \\ &= \cosh u\cos v\end{aligned}$$

$$
\begin{aligned}
\varphi^2 &= Re\Big\{ \int if(1+g^2)\,dz \Big\} \\
&= Re\Big\{ -\frac{i}{2}(e^z - e^{-z}) \Big\} \\
&= Re\Big\{ -i\Big(\frac{e^z - e^{-z}}{2}\Big) \Big\} \\
&= Re\{-i\sinh z\} \\
&= Re\{\cosh u \sin v - i \sinh u \cos v\} \\
&= \cosh u \sin v \\
\varphi^3 &= Re\Big\{ 2\int fg\,dz \Big\} \\
&= Re\Big\{ \int e^z e^{-z}\,dz \Big\} \\
&= Re\{z\} = Re\{u+iv\} = u \\
\implies \phi(u,v) &= (\cosh u \cos v, \cosh u \sin v, u)
\end{aligned}
$$

Which parametrises a catenoid.

This construction is useful but it has a lot of conditions. Consider instead if g is holomorphic with an inverse g^{-1} that is also holomorphic. Set $\tau = g \implies d\tau = g'\,dz$, and define $F(\tau) = f/g'$. Then $F(\tau)\,d\tau = f\,d\tau$. Substitute $g \to \tau$ and $f\,dz \to F(\tau)\,d\tau$ and we get

Theorem 4.32 (Weierstrass-Enneper II)**.** For any holomorphic function $F(\tau)$, a minimal surface is defined by the parametrisation $\phi(z,\bar{z}) = (\varphi^1(z,\bar{z}), \varphi^2(z,\bar{z}), \varphi^3(z,\bar{z}))$ where

$$
\begin{aligned}
\varphi^1 &= Re\Big\{ \int (1-\tau^2)F(\tau)\,d\tau \Big\} \\
\varphi^2 &= Re\Big\{ \int i(1+\tau^2)F(\tau)\,d\tau \Big\} \\
\varphi^3 &= 2Re\Big\{ \int \tau F(\tau) d\tau \Big\}
\end{aligned}
$$

Example 4.33. Let $F(\tau) = (2\tau^2)^{-1}$ and $\tau = e^z$. Then

$$
\begin{aligned}
\varphi^1 &= Re\Big\{ \int (1-\tau^2)F(\tau)\,d\tau \Big\} \\
&= Re\Big\{ \frac{1}{2} \int \frac{1-\tau^2}{\tau^2}\,d\tau \Big\} \\
&= Re\Big\{ \frac{1}{2}(-\tau^{-1} - \tau) \Big\} \\
&= Re\Big\{ -\Big(\frac{e^z + e^{-z}}{2}\Big) \Big\}
\end{aligned}
$$

$$
\begin{aligned}
&= Re\{-\cosh z\}\\
&= Re\{-\cosh u\cos v - i\sinh u\sin v\}\\
&= -\cosh u\cos v\\
\varphi^2 &= Re\Big\{\int i(1+\tau^2)F(\tau)\,d\tau\Big\}\\
&= Re\Big\{\frac{i}{2}\int \frac{1+\tau^2}{\tau^2}\,d\tau\Big\}\\
&= Re\Big\{\frac{i}{2}(-\tau^{-1}+\tau)\Big\}\\
&= Re\Big\{i\Big(\frac{e^z-e^{-z}}{2}\Big)\Big\}\\
&= Re\{i\sinh z\}\\
&= Re\{i\sinh u\cos v - \cosh u\sin v\}\\
&= -\cosh u\sin v\\
\varphi^3 &= 2Re\Big\{\int \tau F(\tau)\,d\tau\Big\}\\
&= Re\Big\{\int \tau^{-1}\,d\tau\Big\}\\
&= Re\{log(\tau)\} = Re\{log(e^z)\}\\
&= Re\{z\} = Re\{u+iv\} = u\\
\implies \phi(u,v) &= (-\cosh u\cos v, -\cosh u\sin v, u)
\end{aligned}
$$

which parametrises a catenoid.

4.5 The Björling Problem

The Weierstrass-Enneper representation allows the construction of a minimal surface from any holomorphic function. This is very useful but we can build on top of it to create minimal surfaces using a more geometric condition.

Definition 4.34. A function $f(x)$ of a real variable x is **real analytic** if $f(z)$ for a complex variable z is holomorphic. Equivalently the taylor series of a real analytic function converges to the function itself. Then $f(z)$ is called the **holomorphic extension** of $f(x)$.

Suppose $\alpha(t) : I \to \mathbb{R}^3$ is a real analytic curve and $\aleph : I \to \mathbb{R}^3$ is a real analytic vector field such that $|\aleph(t)| = 1$ and $\aleph(t) \cdot \alpha'(t) = 0$ for all $t \in I$. Geometrically this is saying that $\aleph$ is orthogonal to the tangent vector of α at all points in I.

The Björling problem asks to then construct a parametrisation $\phi(u, v)$ for a minimal surface M such that

1. The curve α is the restriction of the surface ϕ to $v = 0$, i.e. $\alpha(u) = \phi(u, 0)$ for all $u \in I$.

2. The vector field $\aleph$ agrees with the surface normal N on ϕ along α, i.e. $\aleph(u) = N(u, 0)$ for all $u \in I$.

Since $\alpha(t)$ and $\aleph(t)$ are real analytic, their holomorphic extensions $\alpha(z)$ and $\aleph(z)$ are complex holomorphic functions over $\mathbb{C} \supseteq D \to \mathbb{C}^3$ with $I \subseteq D$.

Theorem 4.35. The only solution to Björling's problem is given by

$$\phi(u, v) = Re\Big\{\alpha(z) - i \int_{u_0}^{z} \aleph(w) \times \alpha'(w)\, dw\Big\} \tag{8}$$

where $u_0 \in I$ is fixed and $z = u + iv \in D$.

We prove this by first assuming some ϕ is a solution to Björling's problem and then showing that ϕ must look like eq. (8). This shows that any solution that may exist must be unique. We then prove existence by showing that a parametrisation like eq. (8) satisfies the conditions of Björling's problem. This proof is from [Opr00].

Proof. Suppose ϕ is a solution to Björling's problem. So ϕ parametrises a minimal surface M. Since isothermal coordinates exist on any minimal surface, assume that ϕ is isothermal.

We have then that $\phi(u, 0) = \alpha(u)$ and $N(u, 0) = \aleph(u)$ for all $u \in I$.

By theorem 4.15, since ϕ is isothermal and minimal, ϕ is harmonic, i.e. $\Delta\phi = \phi_{uu} + \phi_{vv} = 0$. So let φ^j be a harmonic conjugate to ϕ^j such that $\phi^j + i\varphi^j$ is holomorphic. Define the holomorphic function $\beta(z)$ by

$$\begin{aligned} \beta &: D \to \mathbb{C}^3 \\ \beta(z) &= \phi(z) + i\varphi(z) \\ &= \begin{bmatrix} \phi^1 \\ \phi^2 \\ \phi^3 \end{bmatrix} + i \begin{bmatrix} \varphi^1 \\ \varphi^2 \\ \varphi^3 \end{bmatrix} \end{aligned}$$

So then the first derivative of this in terms of u is

$$\begin{aligned} \beta'(z) &= \frac{\partial}{\partial u}(\phi + i\varphi) \\ &= \phi_u + i\varphi_u \end{aligned}$$

Since β is holomorphic it satisfies the Cauchy-Riemann equations. So then

$$\beta'(z) = \phi_u + i\varphi_v = \phi_u - i\phi_v$$

ϕ is isothermal so ϕ_u and ϕ_v are orthogonal. The unit surface normal N is orthogonal to both so we have that $\phi_v = N \times \phi_u$. Therefore

$$\beta'(z) = \phi_u - i(N \times \phi_u)$$

Restricting z to $u \in I$ we get

$$\beta'(u) = \alpha'(u) - i(\aleph(u) \times \alpha'(u))$$

since $\phi(u, 0) = \alpha(u)$ and $N(u, 0) = \aleph(u)$. Integrating this expression in terms of real coordinates yields

$$\beta(u) = \alpha(u) - i \int_{u_0}^{u} \aleph(t) \times \alpha'(t)\, dt$$

Suppose that $\gamma(z)$ is the holomorphic curve defined by

$$\gamma(z) = \alpha(z) - i \int_{u_0}^{z} \aleph(w) \times \alpha'(w)\, dw$$

then we can see that $\beta(u) = \gamma(u + 0i)$ for all $u \in I$ and so by the identity theorem we have that $\beta(z) = \gamma(z)$ for all $z \in D$.

The real part of β is $Re\{\phi + i\varphi\} = \phi$ so

$$\begin{aligned} \phi(u, v) &= Re\{\beta(z)\} \\ &= Re\Big\{\alpha(z) - i \int_{u_0}^{z} \aleph(w) \times \alpha'(w)\, dw\Big\} \end{aligned}$$

which is equivalent to eq. (8) and so we have proved uniqueness.

Now consider the holomorphic function $\beta(z)$ defined by

$$\begin{aligned} \beta(z) &= \phi(z) + i\varphi(z) \\ &= \alpha(z) - i \int_{u_0}^{z} \aleph(w) \times \alpha'(w)\, dw \end{aligned}$$

Take $u \in I$ and note that $\alpha'(u)$ and $\aleph(u)$ are real. Consider the restriction of β to $z = u + 0i$.

$$\begin{aligned} \beta(u) &= \alpha(u) - i \int_{u_0}^{u} \aleph(t) \times \alpha'(t)\, dt \\ \implies \beta'(u) &= \alpha'(u) - i(\aleph(u) \times \alpha'(u)) \end{aligned}$$

and so the real and imaginary parts of β' are

$$
\begin{aligned}
Re\{\beta'(u)\} &= \alpha'(u) \\
Im\{\beta'(u)\} &= -(\aleph(u) \times \alpha'(u))
\end{aligned}
$$

Also note that $\aleph \times \alpha'$ is orthogonal to α' and so $(\aleph \times \alpha') \cdot \alpha' = 0$. Furthermore, since $\aleph$ is orthogonal to α' we have that $|\aleph \times \alpha'| = |\aleph||\alpha'| = |\alpha'|$ since $|\aleph| = 1$.

Now consider

$$
\begin{aligned}
\beta'(u)^2 &= (\alpha'(u) - i(\aleph(u) \times \alpha'(u)))^2 \\
&= \alpha'(u) \cdot \alpha'(u) \\
&\quad - 2i\alpha'(u) \cdot (\aleph(u) \times \alpha'(u)) \\
&\quad - (\aleph(u) \cdot \alpha'(u)) \cdot (\aleph(u) \cdot \alpha'(u)) \\
&= |\alpha'(u)|^2 - 0 - |\aleph(u) \cdot \alpha'(u)|^2 \\
&= |\alpha'(u)|^2 - |\alpha'(u)|^2 \\
&= 0
\end{aligned}
$$

So then since $\beta'(u)^2 = 0$ for all $u \in I$, by the identity theorem $\beta'(z)^2 = 0$ for all $z \in D$. This is precisely the isothermal condition used in the Weierstrass-Enneper representation and so we know that the real part of β parametrises a minimal surface M in isothermal coordinates. This parametrisation is precisely

$$
\begin{aligned}
\phi(u, v) &= Re\{\beta(z)\} \\
&= Re\Big\{\alpha(z) - i \int_{u_0}^{z} \aleph(w) \times \alpha'(w)\, dw\Big\}
\end{aligned}
$$

as before. So we have shown that the surface described by eq. (8) is minimal and in isothermal coordinates. All that is left to show is that conditions 1 and 2 are satisfied.

For 1, since $\aleph(u)$ and $\alpha'(u)$ are real for $u \in I$, $\phi(u, 0) = Re\{\beta(u)\} = \alpha(u)$ and so the curve alpha is the restriction of the surface M to $v = 0$.

For 2, again consider $u \in I$. Recall that $\beta'(u) = \phi_u(u, 0) - i\phi_v(u, 0) = \alpha'(u) - i(\aleph(u) \times \alpha'(u))$. Comparing real and imaginary parts yields

$$
\begin{aligned}
\phi_u(u, 0) &= \alpha'(u) \\
\phi_v(u, 0) &= \aleph(u) \times \alpha'(u)
\end{aligned}
$$

Then ϕ is isothermal $\implies$ ϕ_u and ϕ_v are orthogonal and of equal length so

$$
\begin{aligned}
\phi_v(u,0) &= N(u,0) \times \phi_u(u,0) \\
&= N(u,0) \times \alpha'(u) \\
\implies \aleph(u) \times \alpha'(u) &= N(u,0) \times \alpha'(u) \\
\implies \aleph(u) &= N(u,0)
\end{aligned}
$$

and so the vector field $\aleph$ along α agrees with the surface normal there. □

Lemma 4.36. For a curve $\alpha(t)$, if we select $\aleph(t) = \widehat{N}(t)$ where $\widehat{N}$ is the principle normal curvature vector of α (definition 2.10), α is a geodesic on the resulting minimal surface M under the Björling problem.

Proof. Note that $\aleph(t) = \widehat{N}(t) = N(t)$ so the curvature of α always acts completely in the direction of the surface normal of M. Therefore the geodesic curvature $\kappa_g = 0$ and so α is a straight line on the surface. □

Corollary 4.37 (The Schwarz Reflection Principles)**.** Suppose M is a minimal surface. Then

1. M is symmetric about any straight line contained in M.
2. M symmetric about any plane which intersects M orthogonally.

See [Opr00] for derivation and further discussion.

Corollary 4.38. If $\aleph$ corresponds to the curvature of α, the generated surface will be symmetric about α.

Proof. The result follows immediately from lemma 4.36 and the Schwarz reflection principles. □

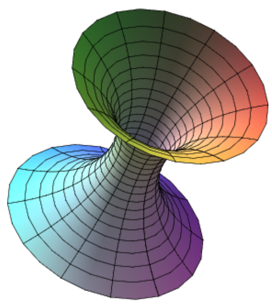
Example 4.39. Consider the curve $\alpha(t)$ and vector field $\aleph(t)$ defined by

$$
\begin{aligned}
\alpha(t) &= (\cos t, \sin t, 0) \\
\aleph(t) = \alpha''(t) &= (-\cos t, -\sin t, 0)
\end{aligned}
$$

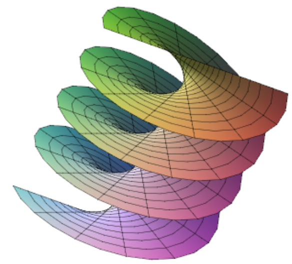
So α will be a geodesic and M will be symmetric about α. The holomorphic extensions we need are given by

$$
\begin{aligned}
\alpha(z) &= (\cos z, \sin z, 0) \\
\alpha'(z) &= (-\sin z, \cos z, 0) \\
\aleph(z) &= (-\cos z, -\sin z, 0) \\
\implies \aleph \times \alpha' &= (0,0,-1)
\end{aligned}
$$

Setting $u_0 = 0$ and integrating the cross product gives



(a) ϕ parametrises a catenoid.



(b) φ parametrises a helcoid.

Figure 7: Plots showing the real and imaginary parts of β.

$$\begin{aligned}\int_0^z (0,0,-1)\,dw &= (0,0,-w)\big|_0^z \\ &= (0,0,-z)\end{aligned}$$

therefore β is

$$\begin{aligned}\beta(z) &= (\cos z, \sin z, 0) - i(0,0,-z) \\ &= (\cos z, \sin z, iz) \\ \beta(u,v) &= (\cos u \cosh v - i \sin u \sinh v, \sin u \cosh v + i \cos u \sinh v, i(u+iv))\end{aligned}$$

and so the minimal surface M is parametrised by

$$\begin{aligned}\phi(u,v) &= Re\{\beta(u,v)\} \\ &= (\cos u \cosh v, \sin u \cosh v, -v)\end{aligned}$$

also recall that

$$\begin{aligned}\varphi(u,v) &= Im\{\beta(u,v)\} \\ &= (-\sin u \sinh v, \cos u \sinh v, u)\end{aligned}$$

These parametrisations are plotted in fig. 7. Notice that the imaginary part of β is a helicoid! This is because the helicoid is the adjoint surface of the catenoid. See [HD10] for further discussion on this.

5 Exotic Minimal Surfaces

In this section we use Maple to render a few interesting minimal surfaces. See [Opr00] for more examples.

Example 5.1. *Sherk's fifth surface* (fig. 8) is parametrised by

$$\phi(u, v) = (\sinh^{-1} u, \sinh^{-1} v, \sinh^{-1} uv)$$

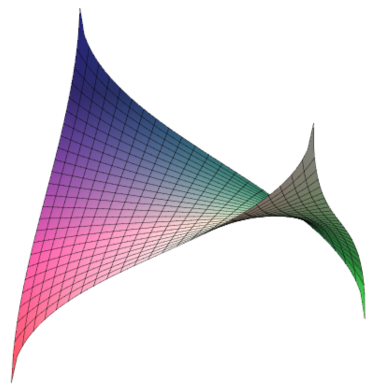


Figure 8: Scherk's fifth surface.

Example 5.2. *Henneberg's surface* (fig. 9) is parametrised by

$$\begin{aligned}\phi(u, v) = (\ &\cosh 2u \cos 2v - 1,\\ &\frac{\sinh 3u \sin 3v}{3} + \sinh u \sin v,\\ &\frac{\sinh 3u \cos 3v}{3} - \sinh u \cos v)\end{aligned}$$

Example 5.3. Take the functions $f = z$, $g = z^3$ and consider the minimal surface defined by ϕ under the Weierstrass-Enneper Representation I (theorem 4.30). This is shown in fig. 10.

Example 5.4. Let $f = g = 1/z^2$ and compute the associate surface M under theorem 4.30, substituting $z \mapsto e^{-iz/2}$ after integrating. Plotting the surface for values of $u \in [0, 4\pi]$ and $v \in [-1/2, 3]$ yields fig. 11.

Example 5.5. We can use Maple to programmatically generate and plot the surfaces associated with a particular curve using the solution to Björling's

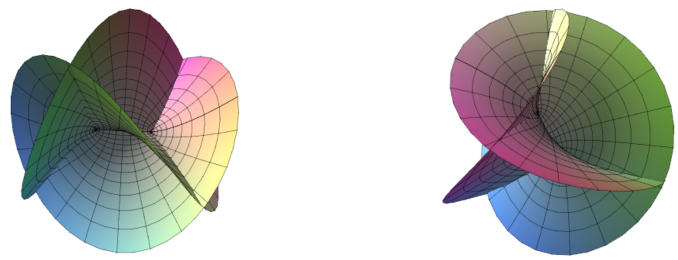


Figure 9: Henneberg's surface.

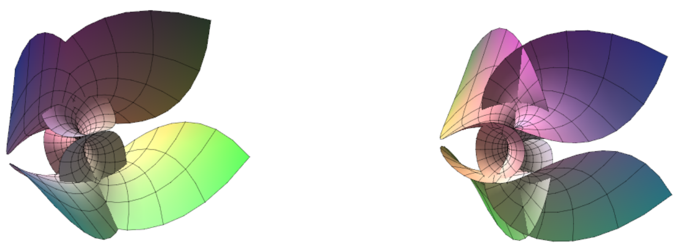


Figure 10: A beautiful minimal surface called "the bat".

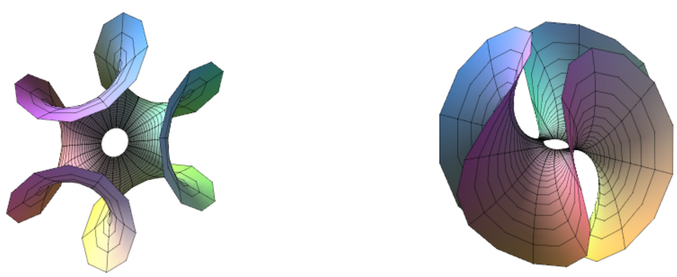


Figure 11: A very curvy minimal surface.

problem. We take an input curve α and set $\aleph$ to the normal principle curvature vector $\widehat{N}$. This set-up means that the curve α is a geodesic on the resulting surface M (lemma 4.36) and M is symmetric about α (corollary 4.38). On these renders, the dark black line corresponds to the original curve α. See [Opr00] for a specific procedure.

Let $\alpha(t) = (1 - \cos t, 0, t - sin(t))$ and plot M for $u \in [0, 4\pi]$ and $v \in [-2, 2]$. The resulting surface is *Catalan's surface* (fig. 12).

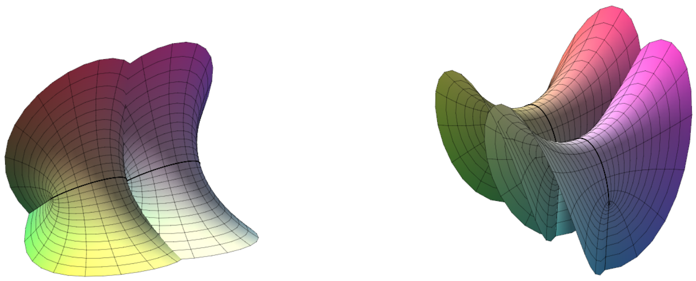


Figure 12: Catalan's surface.

Instead if we take the parabola $\alpha(t) = (t, 0, t^2)$, the resulting minimal surface M is the pringle-like object in fig. 13.

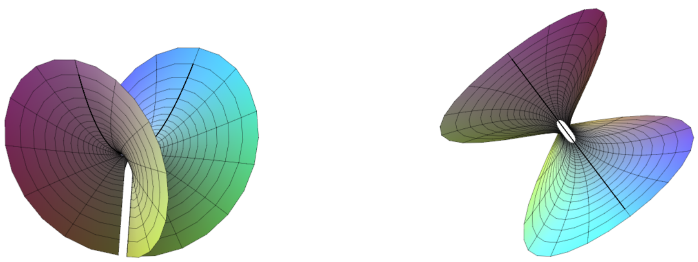


Figure 13: Pringle-like saddle surface generated from a parabola.

Example 5.6. *Enneper's surface* (fig. 14) is generated using $f = 1$, $g = z$ under the Weierstrass-Enneper Representation I (theorem 4.30).

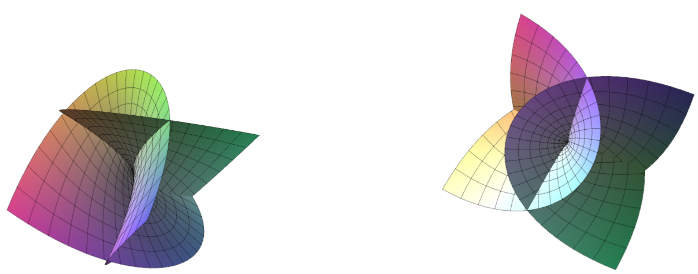


Figure 14: Enneper's surface.

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